Grade 7/8 Math Circles
October 31/November 1/2/3, 2022
Gauss Contests Preparation

About Gauss Mathematics Contests

The Gauss Contests introduce students in Grades 7 and 8 to a broader perspective of mathematics in a fun, accessible way. Intriguing problems and a multiple-choice format make the Gauss Contests a wonderful opportunity for all students in these grades to grow their interest in and get curious about the power of math.

Note
There are Gauss 7 and Gauss 8. Although they might share some questions, they are different contests from each other. If you are in grade 7 or lower, you may choose to write either one of Gauss 7 or Gauss 8.

Contest Information

Audience

- All students in Grades 7 and 8
- Interested students from lower grades

Contest Date

- Ordering (Registration) Deadline: April 25, 2023
- Contest Date: May 17, 2023 (North & South America)

Fees

- $5.00 per participant

Format

- Number of questions: 25 multiple-choice questions
- Duration: 60 minutes
- Score: out of 150
- Format (delivering) of the contest: paper or online
- Some calculators permitted
- Paper dictionaries allowed.

Calculating devices are allowed, provided that they do not have: internet access, the ability to communicate with other devices, information previously stored by students (such as formulas, programs, notes, etc.), a computer algebra system, dynamic geometry software.

Mathematical Content

Questions are based on curriculum common to all Canadian provinces. The last few questions are designed to test ingenuity and insight. Rather than testing content, most of the contest problems test logical thinking and mathematical problem-solving.

Recognition

- A Certificate of Participation is provided for each participant.
- A Certificate of Distinction is provided for each participant scoring in the top 25% of all participants within their own school, for schools with at least 4 participating students.
- A Certificate of Outstanding Achievement is provided to the highest achieving participant in their school on each of the Grade 7 and 8 Contests, for schools with at least 10 participating students.
- The names of some of the top-scoring participants among all those writing the contests are posted online.

Contest Supervisors have the option of generating and printing Participation, Distinction, and School Champion Certificates in our Contest Supervisor Portal.

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Gauss Contest

Grade 8

(The Grade 7 Contest is on the reverse side)

Wednesday, May 18, 2022
(in North America and South America)

Thursday, May 19, 2022
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: 1 hour

Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) information previously stored by students (such as formulas, programs, notes, etc.), (iv) a computer algebra system, (v) dynamic geometry software.

Instructions

1. Do not open the contest booklet until you are told to do so.
2. You may use rulers, compasses and paper for rough work.
3. Be sure that you understand the coding system for your answer sheet. If you are not sure, ask your teacher to explain it.
4. This is a multiple-choice test. Each question is followed by five possible answers marked **A**, **B**, **C**, **D**, and **E**. Only one of these is correct. When you have made your choice, enter the appropriate letter for that question on your answer sheet.
5. Scoring: Each correct answer is worth 5 in Part A, 6 in Part B, and 8 in Part C.
   There is *no penalty* for an incorrect answer.
   Each unanswered question is worth 2, to a maximum of 10 unanswered questions.
6. Diagrams are *not* drawn to scale. They are intended as aids only.
7. When your supervisor instructs you to start, you will have *sixty* minutes of working time.

The name, school and location of some top-scoring students will be published on the website, cemc.uwaterloo.ca. On this website, you will also be able to find copies of past Contests and excellent resources for enrichment, problem solving and contest preparation.

Note
At the time of contest, you will be given a package which contains BOTH Gauss 7 and Gauss 8. One side of the package will be Gauss 7 and the reverse side of the package will be Gauss 8. Please make sure to check that you are writing the correct one.

How to get the most marks out of what you know!

We will list options that will give you the most marks in order, according to the marking instructions:

1. A correct answer for Part C (8 marks)
2. A correct answer for Part B (6 marks)
3. A correct answer for Part A (5 marks)
4. An unanswered question (2 marks) up to 10 questions
5. An incorrect answer for any parts (0 marks)

It is logical that selecting the correct answer for as many questions as possible will gain you the most marks. But, what if you are not sure if the answer is correct or not?

- Fort Part A, a question is worth 5 marks. If you are choosing between 2 options, your probability of getting it right is $\frac{1}{2}$ and $\frac{1}{2} \times 5 = 2.5 > 2$. So you should guess. But if you are choosing between 3 options, you have $\frac{1}{3} \times 5 < 2$. So in this case you should not guess (leave it blank). Similarly, if you are choosing between 4 or 5 options, you should leave the question blank to earn the most marks.
- For Part B, if you are choosing between 4 or 5 options, you are better to leave it blank. If you are choosing between 3 options, you have $\frac{1}{3} \times 6 = 2$, so you can either guess or leave it blank. If you are choosing between 2 options, it is better to guess.
- For Part C, If you are choosing between all 5 options, it is better to leave it blank. If you are choosing between 4 options, either guess or leave it blank. If you are choosing between 2 or 3 options, it is better to guess.

Do not forget to count the number of unanswered questions, since only up to 10 unanswered questions would gain you marks!

Problem Solving Strategies

Some useful strategies when writing Gauss Contests:

1. **Using the information given.** When they give you an information, they are probably expecting you to use that information to solve the problem. Choosing which information to use and the order of using each information are also important when it comes to problem solving.
2. **Working from the 5 possible answers.** Remember, 1 out of 5 possible answers must be correct. You can look at the possible answers first, and rule out the ones that cannot be the correct answer.
3. **Drawing/Using the diagram.** Remember, diagrams are not drawn to scale. Hence, drawing your own diagram could be more helpful than using the one on the question. If the question does not provide a diagram, draw your own! Incorporating some information provided in the question on the diagram will be crucial to problem solving. For example, if they provide a length, indicate that on your diagram! Drawing some lines to further divide your diagram could help you in so many problems.
4. **Looking for patterns.** If they are asking you to find the 2023$^{\text{rd}}$ number in a list, they are NOT expecting you to write down all numbers that come before. Always look for patterns and think about ways to apply that pattern to find your answer!
5. **Working backward.** Since you are given 5 possible answers, you can always try them one by one and find the one that makes sense.

Question 1

The sum of the first 100 positive integers (that is, $1 + 2 + 3 + \cdots + 99 + 100$) equals 5050. The sum of the first 100 positive multiples of 10 (that is, $10 + 20 + 30 + \cdots + 990 + 1000$ equals

(A) 10 100 (B) 5950 (C) 50 500 (D) 6050 (E) 45 450

(Source: 2017 Gauss (Grade 8), #13)

Question 1 Solution

ANSWER: (C)

The question has given us the sum of the first 100 positive integers for a reason! Let's use that information to solve this question.

Since we have $1+2+3+\cdots+99+100=5050$, multiplying both sides by 10 gives us:

$$\begin{aligned} 10\times(1+2+3+\cdots+99+100) &= 10\times 5050 \\ 10+20+30+\cdots+990+1000 &= 50500. \end{aligned}$$

Question 2

The number 503 is a prime number. How many positive integers are factors of 2012?

(A) 2 (B) 3 (C) 7 (D) 6 (E) 8

(Source: 2012 Gauss (Grade 7), #16)

Question 2 Solution

ANSWER: (D)

Before looking at a big number like 2012, why don't we first take a look at how many positive integers are factors of 12? We have

$$12 = 2\times 2\times 3 = 2^2\times 3.$$

Prime factorization is when we write a number in multiples of positive prime numbers like $2,3,5,7,\ldots$ As you have seen above, the prime factorization of 12 is $2^2\times 3$. That is, the positive integers that are factors of 12 are some combinations of 2 or/and 3 in their prime factorizations. Let's say x is a factor of 12. Since 12 contains two 2s (multiplied together), we can choose for x to contain zero, one, or two 2s in the prime factorization of x. Similarly with 3, we can choose for x to contain zero, or one 3 in the prime factorization of x. For instance, if we choose for x to contain two 2s and zero 3 in the prime factorization, then we have $x=2^2\times 3^0=4$ which is a factor of 12. Hence, we have

$$\begin{aligned} (\#\text{ of positive integer factors of }12) &= (\#\text{ of 2s to choose from})\times(\#\text{ of 3s to choose from}) \\ &= 3\times 2 \\ &= 6. \end{aligned}$$

We can now find the prime factorization of 2012. The question gave us that 503 is a prime number. This is because 503 is a factor of 2012. We have

$$\begin{aligned} 2012 &= 503 \times 4 \\ &= 503 \times 2^2. \end{aligned}$$

Hence, we have

$$\begin{aligned} (\#\text{ of positive integer factors of } 2012) &= (\#\text{ of 503s to choose from}) \times (\#\text{ of 2s to choose from}) \\ &= 2 \times 3 \\ &= 6. \end{aligned}$$

Question 3

The values of r, s, t, and u are $2, 3, 4$, and 5, but not necessarily in that order. What is the largest possible value of $r \times s + u \times r + t \times r$?

(A) 24 (B) 45 (C) 33 (D) 40 (E) 49

(Source: 2010 Gauss (Grade 8), #22)

Question 3 Solution

ANSWER: (B)

We first recognize that in the products, $r \times s, u \times r$ and $t \times r$, r is the only variable that occurs in all three. Thus, to make $r \times s + u \times r + t \times r$ as large as possible, we choose $r = 5$, the largest value possible.

Since each of s, u, and t is multiplied by r once only, and the three products are then added, it does not matter which of s, u, or t we let equal $2, 3$, or 4, as the result will be the same.

Therefore, let $s = 2, u = 3$, and $t = 4$.

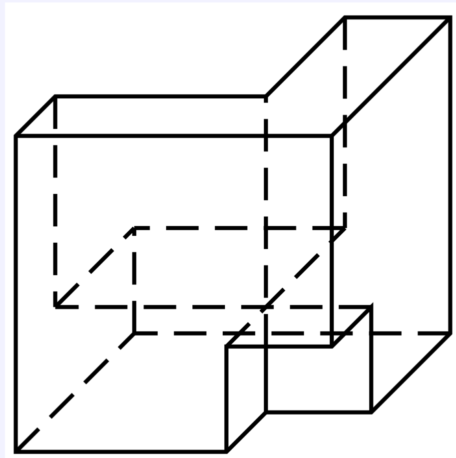
Thus, the largest possible value of $r \times s + u \times r + t \times r$ is $5 \times 2 + 3 \times 5 + 4 \times 5 = 10 + 15 + 20 = 45$.

Question 4

Beginning with a 3 cm by 3 cm by 3 cm cube, a 1 cm by 1 cm by 1 cm cube is cut from one corner and a 2 cm by 2 cm by 2 cm cube is cut from the opposite corner, as shown. In cm^2, what is the surface area of the resulting solid?

(A) 42 (B) 45 (C) 48 (D) 51 (E) 54

(Source: 2014 Gauss (Grade 8), #22)



Question 4 Solution

ANSWER: (E)

Consider the front, bottom right corner of the resulting solid in Figure 1. We now have three new square faces, $RSPQ, RSTU$, and $SPWT$, which were not a part of the surface area of the original cube.

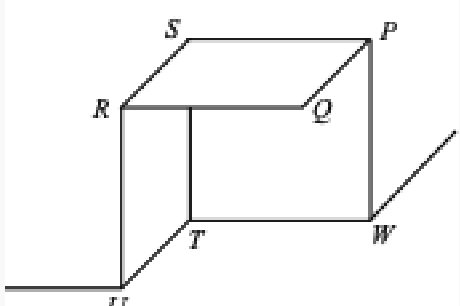


Figure 1

Next, we consider what surface area was part of the original cube that is not present in the resulting solid. To visualize this, we replace the vertex V of the original cube and reconstruct segments QV, UV, and WV, as in Figure 2.

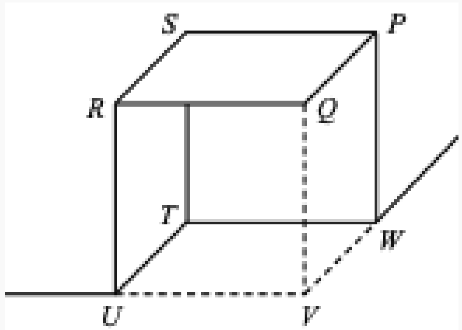


Figure 2

We see that the surface areas that were part of the original cube that are not present in the resulting solid are the 3 square faces $UTWV$, $QPWV$, and $RQVU$. Notice that the sum of the surface areas of these faces are equal to the sum of the surface areas of three new square faces, $RSPQ$, $RSTU$, and $SPWT$.

This is similar for the top left corner of the resulting solid. Hence, the surface area of the resulting solid is the same as the surface area of the original cube which is:

$$3 \times 3 \times 6 = 54.$$

Question 5

The smallest positive integer n for which $n(n+1)(n+2)$ is a multiple of 5 is $n = 3$. All positive integers, n, for which $n(n+1)(n+2)$ is a multiple of 5 are listed in increasing order. What is the 2018^{th} integer in the list?

(A) 3362 (B) 3360 (C) 3363 (D) 3361 (E) 3364

(Source: 2018 Gauss (Grade 8), #23)

Question 5 Solution

ANSWER: (E)

We will first rule out (A) and (D). Let's say we have (A), $n = 3362$. Then $n(n+1)(n+2) = 3362(3363)(3364)$ is not a multiple of 5. Let's say we have (D), $n = 3361$. Then $n(n+1)(n+2) = 3361(3362)(3363)$ is not a multiple of 5.

We will now think about what integers can possibly in this list. We have

$$3, 4, 5, 8, 9, 10, 13, 14, 15, 18, 19, 20, 23, 24, 25, 28, 29, 30, \ldots,$$

to list the first 18 integers in this list. Notice that we have

$$3, 4, 5, 8, 9, 0$$

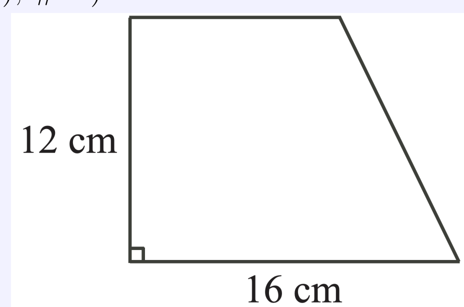
repeating as the last digit of possible n in the list. Since we have 6 options being repeated, we divide 2018 by 6 to get the remainder of 2. That is, $2018 \div 6 = 336 \times 6 + 2 = 2016 + 2$. So the last digit of the 2016^{th} integer in the list is 0. Hence the last digit of the 2018^{th} integer in the list is 4 which leaves us with only (E) as our possible answer.

Question 6

The trapezoid shown has a height of length 12 cm, a base of length 16 cm, and an area of 162 cm^2. The perimeter of the trapezoid is

(A) 51 cm (B) 52 cm (C) $49.\overline{6}$ cm (D) 50 cm (E) 56 cm

(Source: 2011 Gauss (Grade 8), #23)



Question 6 Solution

ANSWER: (B)

The area of a trapezoid can be represented by the formula:

$$h \times \frac{(a+b)}{2}$$

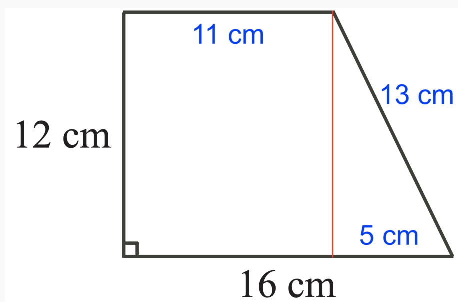
where h is the length of the height, and a and b are the lengths of the bases (two parallel sides).

Let a be the length of the unknown base. Then we have

$$\begin{aligned} 12 \times \frac{(a+16)}{2} &= 162 \\ 6(a+16) &= 162 \end{aligned}$$

Dividing both sides by 6 gives us $a + 16 = 27$. So $a = 11$. To get the length of the remaining unknown side, we use the Pythagorean Theorem. We have

$$\begin{aligned} \sqrt{12^2 + (16-11)^2} &= \sqrt{12^2 + 5^2} \\ &= \sqrt{144 + 25} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$



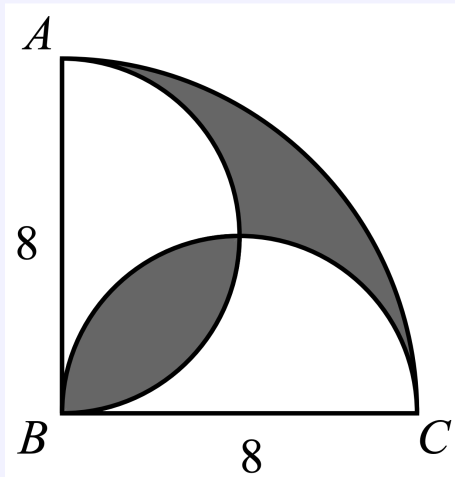
Hence the perimeter of the trapezoid is 12 cm +16 cm +13 cm +11 cm = 52 cm.

Question 7

In the diagram, ABC is a quarter of a circle with radius 8. A semi-circle with diameter AB is drawn, as shown. A second semi-circle with diameter BC is also drawn. The area of the shaded region is closest to

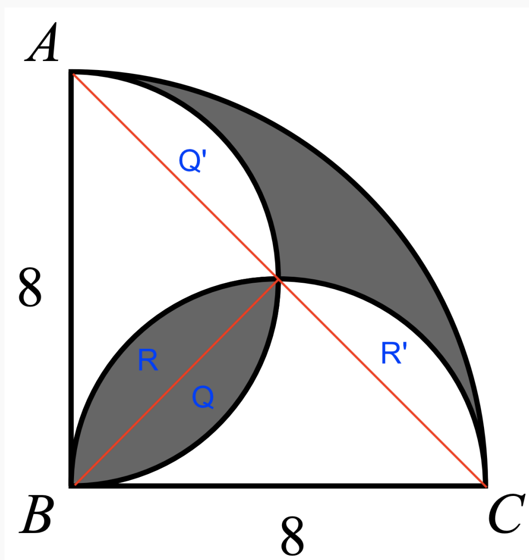
(A) 22.3 (B) 33.5 (C) 25.1 (D) 18.3 (E) 20.3

(Source: 2017 Gauss (Grade 8), #24)

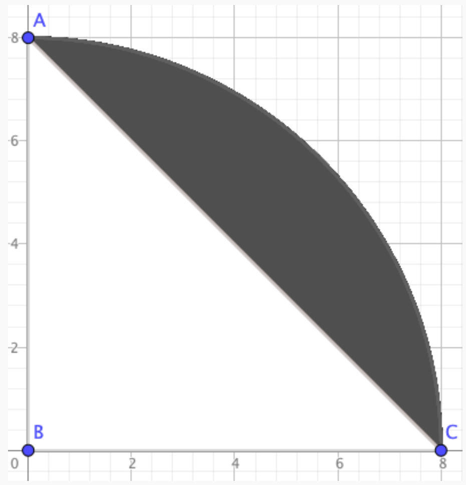


Question 7 Solution

We will draw a straight line connecting A and C, and another straight line dividing the shaded area between the two semi-circles. Those are shown below.



Notice that the area of R is equal to the area of R' and the area of Q is equal to the area of Q'. (To see why the area of region R is equal to the area of region R', in the semicircle that goes through B and C, draw a radius to the intersection point of the two semi-circles, and then compare the two resulting quarter-circles.) So, we can move R (the shaded region) to R' (the unshaded region) and move Q (the shaded region) to Q' (the unshaded region) to get the following diagram.



The area of the shaded region is the area of sector ABC subtracted by the area of the triangle ABC. Hence, the area of the shaded region is

$$\begin{aligned} A &= \pi r^2 \times \frac{1}{4} - \frac{1}{2} \times 8 \times 8 \\ &= \pi (8)^2 \times \frac{1}{4} - 32 \\ &= 16\pi - 32 \\ &\simeq 18.3 \end{aligned}$$

Additional Tips

- Always know the prime factorization of the current year and the previous year!

$$2022 = 2 \times 3 \times 337$$
$$2023 = 7 \times 17^2$$

- Please note that $\cdots$ means that the pattern continues. We have

$$1 + 2 + 3 + \cdots + n = \frac{n \times (n+1)}{2}$$
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n \times (n+1) \times (2n+1)}{6}$$
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n \times (n+1)}{2}\right)^2$$

- Know your Area, Volume, Surface Area, etc. formulas.
- Know squares of each number up to 15.

$$11 \times 11 = 121$$
$$12 \times 12 = 144$$
$$13 \times 13 = 169$$
$$14 \times 14 = 196$$
$$15 \times 15 = 225$$

- Know $2^3 = 8, 2^4 = 16, 2^5 = 32$, and $3^3 = 27, 3^4 = 81$, and $5^3 = 125$.
- Know the Pythagorean Theorem:

$$c^2 = a^2 + b^2$$

  where c is the length of the hypothenuse. Note that $5^2 = 4^2 + 3^2$ and $13^2 = 12^2 + 5^2$.